Vector Representation of an eigenstate

For a set of vectors $\{|1\rangle, |2\rangle, ... |\infty\rangle\}$ that spans the space we are interested in, the arbitrary eigenstate can be decomposed

$$|\psi\rangle = \sum_i^n c_i |i\rangle = \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{pmatrix} \label{1A}$$

The $\{|1\rangle, |2\rangle, ... |\infty\rangle\}$ constitutes a basis (one of many possible) for the space.

Most of the operators we are discussed are linear so

$$\hat{A} |\psi\rangle = \hat{A} \left( \sum_i^n c_i |i\rangle \right) = \sum_i^n c_i \hat{A} |i\rangle \label{2A}\$$

Matrix Representation of an Operator

Operators can be expressed as matrices that "operator" on the eigenvector discussed above

$$\hat{A} |i\rangle = \sum_i^n A_{ij} |i\rangle \label{3A}$$

The number $A_{ij}$ is the $(ij)^{th}$ matrix element of $\hat{A}$ in the basis select.

Hermitian Operators

Hermitian operators are operators that satisfy the general formula

$$\langle \phi_i | \hat{A} | \phi_j \rangle = \langle \phi_j | \hat{A} | \phi_i \rangle \label{Herm1}$$

If that condition is met, then $\hat{A}$ is a Hermitian operator. For any operator that generates a real eigenvalue (e.g., observables), then that operator is Hermitian. The Hamiltonian $\hat{H}$ meets the condition and a Hermitian operator. Equation \ref{Herm1} can be rewritten as

$$A_{ij} = A_{ji}$$

where

$$A_{ij} = \langle \phi_i | \hat{A} | \phi_j \rangle$$

and

$$A_{ji} = \langle \phi_j | \hat{A} | \phi_i \rangle$$

Therefore, when applied to the Hamiltonian operator

$$\boxed{H_{ij}^* = H_{ji}}$$
Multiplication

We can define an inner product (dot product) of two eigenstates $\langle \phi_1 | \phi_2 \rangle$ as

$$\langle \phi_1 | \phi_2 \rangle = \sum_{i=1}^{n} c_i^* c_j$$ \label{4A}

which looks like this in vector representations

$$\langle \phi_1 | \phi_2 \rangle = (c_1^* \; c_2^* \; ... \; c_n^*) \begin{pmatrix} c_1 \ \ c_2 \ \ ... \ \ c_n \end{pmatrix}$$ \label{5A}

This form emphasizes the dot product nature of the inner product multiplication.

From equation \ref{5A}, we can express the bra form of the eigenstate as

$$\langle \phi | = \sum_i^n c_i^* \langle i |$$ \label{6A}

Completeness Relation

For vectors $|i \rangle$ forming an orthonormal basis $\langle i | j \rangle = \delta_{ij}$ for all space then

$$\sum_i^n | i \rangle \langle j | = 1$$ \label{7A}

Those terms in this sum are outer products and are matrices (remember inner products are scalars).

Diagonal Representation of an operator

$$\hat{A} = \sum_i^n \lambda_i | i \rangle \langle i |$$ \label{8A}

This is a matrix that has non-zero element everywhere except the diagonal. E.g.,

$$\begin{pmatrix} 4 & 0 & 0 \ \ 0 & -1 & 0 \ \ 0 & 0 & -4 \end{pmatrix}$$