Matrices can be used to map one set of coordinates or functions onto another set. Matrices used for this purpose are called *transformation matrices*. In group theory, we can use transformation matrices to carry out the various symmetry operations considered at the beginning of the course. As a simple example, we will investigate the matrices we would use to carry out some of these symmetry operations on a vector $\begin{pmatrix} x, y \end{pmatrix}$.

### The identity Operation

The identity operation leaves the vector unchanged, and as you may already suspect, the appropriate matrix is the identity matrix.

$$\begin{pmatrix} x, y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x, y \end{pmatrix}$$  \[\text{(9.1)}\]

### Reflection in a plane

The simplest example of a reflection matrix corresponds to reflecting the vector $\begin{pmatrix} x, y \end{pmatrix}$ in either the $x$ or $y$ axes. Reflection in the $x$ axis maps $y$ to $-y$, while reflection in the $y$ axis maps $x$ to $-x$.

- Reflection in the $x$ axis transforms the vector $\begin{pmatrix} x, y \end{pmatrix}$ to $\begin{pmatrix} x, -y \end{pmatrix}$, and the appropriate matrix is
  $$\begin{pmatrix} x, y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} x, -y \end{pmatrix}$$  \[\text{(9.2)}\]

- Reflection in the $y$ axis transforms the vector $\begin{pmatrix} x, y \end{pmatrix}$ to $\begin{pmatrix} -x, y \end{pmatrix}$, and the appropriate matrix is
  $$\begin{pmatrix} x, y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -x, y \end{pmatrix}$$  \[\text{(9.3)}\]

*Figure (PageIndex{1})*: Reflection across the $x$-axis

Reflection in the $y$ axis transforms the vector $\begin{pmatrix} x, y \end{pmatrix}$ to $\begin{pmatrix} -x, y \end{pmatrix}$, and the appropriate matrix is

$$\begin{pmatrix} x, y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -x, y \end{pmatrix}$$  \[\text{(9.3)}\]
More generally, matrices can be used to represent reflections in any plane (or line in 2D). For example, reflection in the 45° axis shown below maps

\[
\begin{pmatrix} x, y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -y, -x \end{pmatrix}
\]

\label{9.4}

Figure \(\PageIndex{2}\): Reflection across the y-axis

Rotation about an Axis

In two dimensions, the appropriate matrix to represent rotation by an angle \(\theta\) about the origin is

\[
R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
\]

\label{9.5}

In three dimensions, rotations about the \(x\), \(y\) and \(z\) axes acting on a vector \(\begin{pmatrix} x, y, z \end{pmatrix}\) are represented by the following matrices.

\[
R_{x}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}
\]

\label{9.6a}

\[
R_{y}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}
\]

\label{9.6b}

\[
R_{z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\label{9.6c}

Figure \(\PageIndex{3}\): Reflection across the axis that is rotated 45° with respect to x-axis.
Contributors

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