Wien's displacement law states that the blackbody radiation curve for different temperatures peaks at a wavelength inversely proportional to the temperature. The shift of that peak is a direct consequence of the Planck radiation law which describes the spectral brightness of black body radiation as a function of wavelength at any given temperature. However, it had been discovered by Wilhelm Wien several years before Max Planck developed that more general equation, and describes the entire shift of the spectrum of black body radiation toward shorter wavelengths as temperature increases.

Derive Wien's displacement law from Planck's law. Proceed as follows:

\[
\rho (\nu, T) = \frac{2 h \nu^3}{c^3\left(e^{\frac{h\nu}{k_B T}}-1\right)} \tag{Planck2}
\]

We need to evaluate the derivative of Equation \ref{Planck2} with respect to \((\nu)\) and set it equal to zero to find the peak wavelength.

\[
\frac{d}{d\nu} \left\{ \rho (\nu, T) \right\} = \frac{d}{d\nu} \left\{ \frac{2 h \nu^3}{c^3\left(e^{\frac{h\nu}{k_B T}}-1\right)} \right\} = 0 \tag{eq2}
\]

This can be solved via the quotient rule or product rule for differentiation. Selecting the latter for convenience requires rewriting Equation \ref{eq2} as a product:

\[
\left\{ \frac{d}{d\nu} \left\{ \rho (\nu, T) \right\} \right\} = \frac{2h}{c^3} \frac{d}{d\nu} \left\{ ( \nu^3) \left( e^{\frac{h\nu}{k_B T}}-1\right)^{-1} \right\} = 0
\]

applying the product rule (and power rule and chain rule)

\[
\left\{ \frac{d}{d\nu} \left\{ (3 \nu^2) \left( e^{\frac{h\nu}{k_B T}}-1\right)^{-1} - ( \nu^{3}) \left( e^{\frac{h\nu}{k_B T}}-1\right)^{-2} \left(\frac{h}{k_BT}\right) e^{\frac{h\nu}{k_B T}} \right\} = 0\right\}
\]

so this expression is zero when

\[
(3 \nu^2) \left( e^{\frac{h\nu}{k_B T}}-1\right) - ( \nu^{3}) \left( e^{\frac{h\nu}{k_B T}}-1\right)^{-2} \left(\frac{h}{k_BT}\right) e^{\frac{h\nu}{k_B T}} = 0 \tag{eq10}
\]

We can do a substitution \(u = \frac{h\nu}{k_B T}\) and Equation \ref{eq10} becomes

\[
3 (e^u - 1) - u e^u = 0
\]

Finding the solutions to this equation requires using Lambert's W-functions and results numerically in

\[
u = 3 + W(-3e^{-3}) \approx 2.8214
\]

so unsubstituting the \((u)\) variable

\[
u = \frac{h\nu}{k_B T} \approx 2.8214 \tag{eq20}
\]
or

\[ \nu \approx \frac{2.8214 \cdot k_B}{h} T \approx \frac{(2.8214 \cdot (1.38 \times 10^{-23} \text{ J/K}) \cdot 6.63 \times 10^{-34} \text{ J/s})}{T} \approx (5.8 \times 10^{10} \text{ Hz/K}) T \]

The consequence is that the shape of the blackbody radiation function would shift proportionally in frequency with temperature. When Max Planck later formulated the correct blackbody radiation function it did not include Wien's constant explicitly. Rather, Planck's constant \( h \) was created and introduced into his new formula. From Planck's constant \( h \) and the Boltzmann constant \( k \), Wien's constant (Equation \ref{eq20}) can be obtained.

Contributors

- Wikipedia