The expected value (or expectation, mathematical expectation, mean, or first moment) refers to the value of a variable one would "expect" to find if one could repeat the random variable process an infinite number of times and take the average of the values obtained. More formally, the expected value is a weighted average of all possible values.

### Expectation value of momentum

\[
\langle p_z \rangle = \int \Psi^* p_z \Psi \, dz
\]

where

\[
p_z = -i \hbar \frac{\partial}{\partial z}
\]

Example \((\PageIndex{1})\): Free Particles

For a free particle \(\Psi = e^{ikz}\).

\[
-\hbar \frac{\partial}{\partial z} e^{ikz} = \hbar k e^{ikz}
\]

Example \((\PageIndex{2})\): Bound in a Box

For a bound particle in a box of length \(L\), \(\Psi = \sqrt{(2/L)} \sin(n\pi z/L)\)

\[
\langle p_z \rangle = \frac{2}{L} \left(-i \frac{n\pi \hbar}{L}\right) \int_0^L \sin \left(\frac{n\pi z}{L}\right) \cos \left(\frac{n\pi z}{L}\right) dz
\]

Following a simple substitution approach to solving this integral, let \((u = n\pi z/L)\) and \((du = n\pi/L \, dz)\), then \((dz = L/n\pi du)\)

\[
\langle p_z \rangle = \frac{2}{L} \left(-i \hbar\right) \int_0^{n\pi} \sin(u) \cos(u) \, du = 0
\]

Note that this makes sense since the particles spends an equal amount of time traveling in the \(+x\) and \(-x\) direction.

### Expectation value of energy

\[
\langle E_z \rangle = \int \Psi^* T_z \Psi \, dz
\]

where the translation energy operation in the z-direction:

\[
[T_z = -\hbar^2 \frac{\partial^2}{\partial z^2}]
\]

For a bound particle

\[
-\hbar^2 \frac{\partial^2}{\partial z^2} \sin \left(\frac{n\pi z}{L}\right) = \frac{\hbar^2 n^2 \pi^2}{2mL^2} \sin
\]
and using the normalized wavefunction

\[ \langle E_z \rangle = \frac{2}{L} \left( \frac{\hbar^2 n^2 \pi^2}{2mL^2} \right) \int_0^L \sin^2 \left( \frac{n \pi z}{L} \right) dz \]

Let \( u = \frac{npz}{L} \) and \( du = \frac{np}{L} \, dz \), then \( dz = \frac{L}{np} \, du \)

\[ \langle E_z \rangle = \frac{2}{L} \left( \frac{\hbar^2 n^2 \pi^2}{2mL^2} \right) \left( \frac{L}{n \pi} \right) \int_0^{n \pi} \sin^2(u) \, du = \frac{2}{L} \left( \frac{\hbar^2 n^2 \pi^2}{2mL^2} \right) \left( \frac{L}{n \pi} \right) \frac{u}{2} \biggr\vert_0^{n \pi} = \frac{\hbar^2 n^2 \pi^2}{2mL^2} = \frac{\hbar^2 n^2}{8mL^2} \]

**Expectation value of position**

\[ \langle z \rangle = \int \Psi^* z \Psi \, dz \]

For the bound particle in a box.

\[ \langle z \rangle = \frac{2}{L} \int_0^L \sin^2 \left( \frac{n \pi z}{L} \right) z \, dz \]

Let \( u = \frac{npz}{L} \) and \( du = \frac{np}{L} \, dz \), then \( dz = \frac{L}{np} \, du \)

\[ \langle z \rangle = \frac{2}{L} \left( \frac{L^2}{n^2 \pi^2} \right) \int_0^{n \pi} \sin^2(u) \, u \, du = \frac{2}{L} \left( \frac{L^2}{n^2 \pi^2} \right) \left[ \frac{u^2}{4} + \frac{\sin^2(u)}{4} - \frac{\cos(u) \sin(u)}{2} \right] \biggr\vert_0^{n \pi} \]

Therefore, \( \langle z \rangle = L/2 \). This is logical since the average position of the particle is in the middle of the box.

**Contributors and Attributions**

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