Recap of Lecture 11

Last lecture continued the discussion of vibrations into the realm of quantum mechanics. We reviewed the classical picture of vibrations including the classical potential, bond length, and bond energy. We then introduced the quantum version using the harmonic oscillator as an approximation of the true potential. This involves constructing a Hamiltonian with a parabolic potential. Solving the resulting (time-independent) Schrödinger equation to obtain the eigenstates, energies, and quantum numbers \( v \) results is beyond this course, so they are given. Key aspect of these solution are the fundamental frequency and zero-point energy.

Recap of Lecture 10

Last lecture addressed three aspects. The first is the introduction of the commutator which is meant to evaluate is two operators commute (a property used extensively in basic algebra courses). Not every pair of operators will commute meaning the order of operations matter. The second aspect is to redefine the Heisenberg Uncertainty Principle now within the context of commutators. Now, we can identify if any two quantum measurements (i.e., eigenvalues of specific operators) will require the Heisenberg Uncertainty Principle to be addressed when simultaneously evaluating them. The third aspect of the lecture was the introduction of vibrations, including how many vibrations a molecule can he (depending on linearity) and the origin of this. The solutions to the harmonic oscillator potential were qualitatively shown (via Java application) with an emphasis of the differences between this model system and the particle in the box (important).

<table>
<thead>
<tr>
<th></th>
<th>Atoms (very symmetric)</th>
<th>Linear molecules (less symmetric)</th>
<th>Non-linear molecules (most unsymmetric)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translation (x, y, and z)</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Rotation (x, y, and z)</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Vibrations</td>
<td>0</td>
<td>3(N - 5)</td>
<td>3(N - 6)</td>
</tr>
<tr>
<td>Total (including Vibration)</td>
<td>3</td>
<td>3(N)</td>
<td>3(N)</td>
</tr>
</tbody>
</table>

Quantum Mechanical Vibrations

The simplified potential discussed in general chemistry is the potential energy function used in constructing the Hamiltonian. From solving the Schrödinger equations, we get eigenfunctions, eigenvalues (energies) and quantum numbers. Combining these on the potential like we did for the particle in a box give a more detailed (quantum) picture
Zero-point energy

Zero-point energy is the lowest possible energy that a quantum mechanical system may have, i.e. it is the energy of the system's ground state. The uncertainty principle states that no object can ever have precise values of position and velocity simultaneously.

For the particle in the 1D box of length $L$, with energies

$$E_n = \dfrac{h^2 n^2}{8mL^2}$$

the zero-point energy ($n=1$) is

$$E_{ZPE} = \dfrac{h^2}{8mL^2}$$

For the Harmonic Oscillator with energies

$$E_v = \left(v + \dfrac{1}{2}\right)h\nu$$

the zero-point energy ($v=0$) is

$$E_v = \dfrac{h\nu}{2} = \dfrac{\hbar\omega}{2}$$

Many people have made cookie ideas of tapping into the zero-point energy to drive our economy. This is a silly idea and impossible, since this energy cannot never be tapped (see https://van.physics.illinois.edu/qa/index.php?id=1256 for more discussion)

Infrared Spectroscopy

Infrared (IR) spectroscopy is one of the most common and widely used spectroscopic techniques employed mainly by inorganic and organic chemists due to its usefulness in determining structures of compounds and identifying them.

Chemical compounds have different chemical properties due to the presence of different functional groups. A molecule
composed of \(n\)-atoms has \(3n\) degrees of freedom, six of which are translations and rotations of the molecule itself. This leaves \(3N-6\) degrees of vibrational freedom (\(3N-5\) if the molecule is linear). Vibrational modes are often given descriptive names, such as stretching, bending, scissoring, rocking and twisting. The four-atom molecule of formaldehyde, the gas phase spectrum of which is shown below, provides an example of these terms.

The spectrum of gas phase formaldehyde, is shown below.

![Gas Phase Infrared Spectrum of Formaldehyde, \(H_2C=O\)](Image)

**Characteristic normal modes (vibrations) in formaldehyde**

- \(\text{CH}_2\) Asymmetric Stretch
- \(\text{CH}_2\) Symmetric Stretch
- \(\text{C} = \text{O}\) Stretch
- \(\text{CH}_2\) Scissoring
- \(\text{CH}_2\) Rocking
- \(\text{CH}_2\) Wagging

The exact frequency at which a given vibration occurs is determined by the strengths of the bonds involved and the mass of the component atoms.

**Transition Moment Integrals gives Selection rules**

Spectroscopy is a matter-light interaction. You first need to know the results of the Schrödinger equation of a specific system. This include both eigenstates (wavefuctions), eigenvalues (energies), and quantum numbers and You need to understand how to couple the eigenstates with electromagnetic radiation. This is done via the *transition moment integral*:

\[
\langle \psi_i | \hat{M} | \psi_f \rangle
\]
The transition moment integral gives information about the probability of a transition occurring. For IR of a single harmonic oscillator, \( \langle \hat{M} \rangle \) can be set to \( \langle x \rangle \). A more detailed discussion will be presented later. So the probability for a transition in HO is:

\[
P_{i \rightarrow f} = \langle \psi_i | x | \psi_f \rangle
\]

- From Equation \( \ref{26} \) comes general rules for absorption. For IR, the transition is allowed only if the molecule has a changing dipole moment.
- From Equation \( \ref{26} \) comes selection rules (what possible transitions are allowed). For IR this results in \( \Delta v = \pm 1 \).

The vibration must change the molecular dipole moment to have a non-zero (electric) transition dipole moment. Molecules CAN have a zero net dipole moment, yet STILL UNDERGO transitions when stimulated by infrared light.

### Dipole Moments (rehash from gen chem)

When two electrical charges, of opposite sign and equal magnitude, are separated by a distance, a dipole is established. The size of a dipole is measured by its dipole moment \( \langle \mu \rangle \). Dipole moment is measured in Debye units, which is equal to the distance between the charges multiplied by the charge (1 Debye equals \( 3.34 \times 10^{-30} \ C \, m \)). The dipole moment of a molecule can be calculated by Equation \( \ref{d1} \):

\[
\vec{\mu} = \sum_i q_i \, \vec{r}_i \tag{d1}
\]

where

- \( \langle \vec{\mu} \rangle \) is the dipole moment vector
- \( \langle q_i \rangle \) is the magnitude of the \( \langle i^{th} \rangle \) charge, and
- \( \langle \vec{r}_i \rangle \) is the vector representing the position of \( \langle i^{th} \rangle \) charge.

The dipole moment acts in the direction of the vector quantity. An example of a polar molecule is \( \langle H_2O \rangle \). Because of the lone pair on oxygen, the structure of H\(_2\)O is bent (via VEPSR theory), which that the vectors representing the dipole moment of each bond do not cancel each other out. Hence, water is polar.

![Dipole moment of water. The convention in chemistry is that the arrow representing the dipole moment goes from positive to negative. In physics, the opposite is used.](image)

Example \( \langle \text{PageIndex(1)} \rangle \):
Which molecules absorb IR radiation?

\( \text{O}_2 \)
**Answer:**
No: Vibration does not change the dipole moment of the molecule due to symmetry.

\( \text{HCl} \)
**Answer:**
Yes: Vibration does change the dipole moment of the molecule since there is a difference in electronegativity so the distance between the two atoms affects the dipole moment (Equation \ref{d1}).

\( \text{H}_2 \)
**Answer:**
No: Vibration does not change the dipole moment of the molecule due to symmetry.

\( \text{Cl}_2 \)
**Answer:**
No: Vibration does not change the dipole moment of the molecule due to symmetry.

\( \text{CO}_2 \)
**Answer:**
Yes: A vibration does change the dipole moment of the molecule since there is a difference in electronegativity so the distance between the two atoms affects the dipole moment. This is not the symmetric stretch, but the other modes.

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**Fig. 1 Vibrational modes of CO\(_2\)**
Using the harmonic oscillator and wave equations of quantum mechanics, the energy can be written in terms of the spring constant and reduced mass as

\[ E = (v + \frac{1}{2}) \ \frac{h}{2\pi} \sqrt{\frac{k}{\mu}} \]  \hspace{1cm} \text{(16)}

where \( h \) is Planck’s constant and \( v \) is the vibrational quantum number and ranges from 0, 1, 2, 3, … infinity.

\[ E = (v + \frac{1}{2}) h \nu \]  \hspace{1cm} \text{(17)}

where \( \nu \) is the vibrational frequency. Transitions in vibrational energy levels can be brought about by absorption of radiation, provided the energy of the radiation exactly matches the difference in energy levels between the vibrational quantum states and provided the vibration causes a change in dipole moment. This can be expressed as

\[ \Delta E = h \nu = \frac{h}{2\pi} \sqrt{\frac{k}{\mu}} \]  \hspace{1cm} \text{(18)}

At room temperature, the majority of molecules are in the ground state \( v = 0 \), from the equation above

\[ E_0 = \frac{1}{2} h \nu \]  \hspace{1cm} \text{(19)}

when a molecule absorbs energy, there is a promotion to the first excited state

\[ E_1 = \frac{3}{2} h \nu \]  \hspace{1cm} \text{(20)}

\[ \left( \frac{3}{2} h \nu_m - \frac{1}{2} h \nu_v \right) = h \nu \]  \hspace{1cm} \text{(21)}

The frequency of radiation \( \nu \) that will bring about this change is identical to the classical vibrational frequency of the bond and it can be expressed as

\[ E_{\text{radiation}} = h \nu = \Delta E = h \nu_m = \frac{h}{2\pi} \sqrt{\frac{k}{\mu}} \]  \hspace{1cm} \text{(22)}

The above equation can be modified so that the radiation can be expressed in wavenumbers

\[ \tilde{\nu} = \frac{1}{2\pi c} \sqrt{\frac{k}{\mu}} \]  \hspace{1cm} \text{(23)}

where

- \( c \) is the velocity of light \( (\text{cm s}^{-1}) \) and
- \( \tilde{\nu} \) is the wave number of an absorption maximum \( (\text{cm}^{-1}) \)}
Selection Rules

Photons can be absorbed or emitted, and the harmonic oscillator can go from one vibrational energy state to another. Which transitions between vibrational states are allowed? If we take an infrared spectrum of a molecule, we see numerous absorption bands, and we need to relate these bands to the allowed transitions involving different normal modes of vibration.

The selection rules are determined by the transition moment integral.

\[
\mu_T = \int_{-\infty}^{\infty} \Psi_{v'}^*(Q) \hat{\mu}(Q) \Psi_v(Q) dQ = \langle \Psi_{v'} | \hat{\mu} | \Psi_v \rangle \tag{6.6.1}
\]

To evaluate this integral we need to express the dipole moment operator, \(\hat{\mu}\), in terms of the magnitude of the normal coordinate \(Q\). Evaluating the integral in Equation (6.6.1) can be difficult depending on the complexity of the wavefunctions used. We can often (although not always) take advantage of the symmetries of the wavefunction (and \(\hat{\mu}\) too) to make things easier.
Figure: (left) \( f(x) = x^2 \) is an example of an even function. (right) \( f(x) = x^3 \) is an example of an odd function. Images used with permission from Wikipedia.

While functions exhibit this symmetry, the product of functions inherent the symmetries of the constituent components via a "product table." The one below is in term of odd/even symmetry, but as you will learn in other classes (especially
group theories), there are several symmetries that 3D objects have. The product tables constructed that take into account all such symmetries are more complicated.

<table>
<thead>
<tr>
<th>Product table</th>
<th>Odd Function (anti-symmetric)</th>
<th>Even Function (symmetric)</th>
<th>No symmetry (neither odd nor even)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Odd Function (anti-symmetric)</td>
<td>Even Function (symmetric)</td>
<td>Odd Function (anti-symmetric)</td>
<td>who knows</td>
</tr>
<tr>
<td>Even Function (symmetric)</td>
<td>Odd Function (anti-symmetric)</td>
<td>Even Function (symmetric)</td>
<td>who knows</td>
</tr>
<tr>
<td>No symmetry (neither odd nor even)</td>
<td>who knows</td>
<td>who knows</td>
<td>who knows</td>
</tr>
</tbody>
</table>

These symmetries are important since the integrand of an integral (over all space) of an odd function is ALWAYS zero. So you do no need to solve it.

And now the Harmonic Oscillator Wavefunctions

Because of the association of the wavefunction with a probability density, it is necessary for the wavefunction to include a normalization constant, \( N_v \).

\[
N_v = \frac{1}{(2^v v! \sqrt{\pi})^{1/2}} \tag{5.6.15}
\]

The final form of the harmonic oscillator wavefunctions is the product of thee terms:

\[
| \Psi_v(x) \rangle = N_v H_v(x) e^{-x^2/2} \tag{5.6.16}
\]

The first few physicists' Hermite polynomials are:

\[
\begin{align*}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
\end{align*}
\]

The symmetry of the Harmonic Oscillator wavefunctions is dictated by the Hermite Polynomials.
• Which levels are even functions and which are odd functions?
• Without calculating an integral, what is $\langle x \rangle$?

Contributors

• Adapted from "Quantum States of Atoms and Molecules" by David M. Hanson, Erica Harvey, Robert Sweeney, Theresa Julia Zielinski