Stirling's approximation is named after the Scottish mathematician James Stirling (1692-1770). In confronting statistical problems we often encounter factorials of very large numbers. The factorial $N!$ is a product $N(N-1)(N-2)..(2)(1)$. Therefore, $\ln(N!)$ is a sum

$$\ln N! = \ln 1 + \ln 2 + \ln 3 + ... + \ln N = \sum_{k=1}^N \ln k. \tag{1}$$

where we have used the property of logarithms that $\log(abc) = \log(a) + \log(b) + \log(c))$. The sum is shown in figure below.

![Graph showing the approximation of $\ln(N!)$](image)

Using Euler-MacLaurin formula one has

$$\sum_{k=1}^N \ln k = \int_1^N \ln x \, dx + \sum_{k=1}^p \frac{B_{2k}}{2k(2k-1)} \left( \frac{1}{n^{2k-1}} - 1 \right) + R, \tag{2}$$

where $B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, ...$ are the Bernoulli numbers, and $R$ is an error term which is normally small for suitable values of $p$.

Then, for large $N$,

$$\ln N! \sim \int_1^N \ln x \, dx \approx N \ln N - N. \tag{3}$$

after some further manipulation one arrives at (apparently Stirling's contribution was the prefactor of $\sqrt{2\pi}$)

$$N! = \sqrt{2\pi N} \, N^{N} e^{-N} e^{\lambda_N} \tag{4}$$

where

$$\frac{1}{12N+1} < \lambda_N < \frac{1}{12N}. \tag{5}$$

The sum of the area under the blue rectangles shown below up to $N$ is $\ln N!$. As you can see the rectangles begin to closely approximate the red curve as $m$ gets larger. The area under the curve is given the integral of $\ln x$.

$$\ln N! = \sum_{m=1}^N \ln m \approx \int_1^N \ln x \, dx \tag{6}$$

To solve the integral use integration by parts

$$\int u \, dv = uv - \int v \, du \tag{7A}$$
Here we let \( u = \ln x \) and \( dv = dx \). Then \( v = x \) and \( du = \frac{dx}{x} \).

\[
\int_{0}^{N} \ln x \, dx = x \ln x|_0^N - \int_0^N x \frac{dx}{x} \tag{7B}
\]

Notice that \( x/x = 1 \) in the last integral and \( x \ln x \) is 0 when evaluated at zero, so we have

\[
\int_{0}^{N} \ln x \, dx = N \ln N - \int_0^N dx \tag{8}
\]

Which gives us Stirling’s approximation: \( \ln N! = N \ln N – N \). As is clear from the figure above Stirling’s approximation gets better as the number \( N \) gets larger (Table \( \PageIndex{1} \)).

Table \( \PageIndex{1} \): Evaluation of Approximation with absolute values

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N! )</th>
<th>( \ln N! )</th>
<th>( N \ln N – N )</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>( 3.63 \times 10^6 )</td>
<td>15.1</td>
<td>13.02</td>
<td>13.8%</td>
</tr>
<tr>
<td>50</td>
<td>( 3.04 \times 10^{64} )</td>
<td>148.4</td>
<td>145.6</td>
<td>1.88%</td>
</tr>
<tr>
<td>100</td>
<td>( 9.33 \times 10^{157} )</td>
<td>363.7</td>
<td>360.5</td>
<td>0.88%</td>
</tr>
<tr>
<td>150</td>
<td>( 5.71 \times 10^{262} )</td>
<td>605.0</td>
<td>601.6</td>
<td>0.56%</td>
</tr>
</tbody>
</table>

Calculators often overheat at 200!, which is all right since clearly result are converging. In thermodynamics, we are often dealing very large \( N \) (i.e., of the order of Avagadro’s number) and for these values Stirling’s approximation is excellent.

References


Contributors and Attributions

- SklogWiki

Stefan Franzen (North Carolina State University)