Stirling's approximation is named after the Scottish mathematician James Stirling (1692-1770). In confronting statistical problems we often encounter factorials of very large numbers. The factorial $N!$ is a product $N(N-1)(N-2)\ldots(2)(1)$. Therefore, $\ln(N!)$ is a sum

$$\ln(N!) = \ln 1 + \ln 2 + \ln 3 + \ldots + \ln N = \sum_{k=1}^{N} \ln k.$$  \label{1}

where we have used the property of logarithms that $\log(abc) = \log(a) + \log(b) + \log(c))$. The sum is shown in figure below.

![Graph of ln(x) vs x](image)

Using Euler-MacLaurin formula one has

$$\sum_{k=1}^{N} \ln k = \int_1^{N} \ln x \, dx + \sum_{k=1}^{p} \frac{B_{2k}}{2k(2k-1)} \left(\frac{1}{n^{2k-1}} - 1\right) + R,$$  \label{2}

where $B_{1} = -1/2$, $B_{2} = 1/6$, $B_{3} = 0$, $B_{4} = -1/30$, $B_{5} = 0$, $B_{6} = 1/42$, $B_{7} = 0$, $B_{8} = -1/30$, ... are the Bernoulli numbers, and $R$ is an error term which is normally small for suitable values of $p$.

Then, for large $N$,

$$\ln N! \sim \int_1^{N} \ln x \, dx \approx N \ln N - N.$$  \label{3}

after some further manipulation one arrives at (apparently Stirling's contribution was the prefactor of $\sqrt{2\pi}$)

$$N! \approx \sqrt{2\pi N} \, N^N \, e^{-N} \, e^{\lambda_N},$$  \label{4}

where

$$\frac{1}{12N+1} < \lambda_N < \frac{1}{12N}.$$  \label{5}

The sum of the area under the blue rectangles shown below up to $N$ is $\ln N!$. As you can see the rectangles begin to closely approximate the red curve as $m$ gets larger. The area under the curve is given the integral of $\ln x$.

$$\ln N! = \sum_{m=1}^{N} \ln m \approx \int_1^{N} \ln x \, dx,$$  \label{6}

To solve the integral use integration by parts

$$\int u \, dv = uv - \int v \, du.$$  \label{7A}
Here we let \(u = \ln x\) and \(dv = dx\). Then \(v = x\) and \(du = \frac{dx}{x}\).

\[
\int_0^N \ln x \, dx = \int_0^N x \frac{dx}{x} \tag{7B}
\]

Notice that \(x/x = 1\) in the last integral and \((x \ln x)\) is 0 when evaluated at zero, so we have

\[
\int_0^N \ln x \, dx = N \ln N - \int_0^N dx \tag{8}\]

Which gives us Stirling’s approximation: \(\ln N! = N \ln N – N\). As is clear from the figure above Stirling’s approximation gets better as the number N gets larger (Table \(\PageIndex{1}\)).

Table \(\PageIndex{1}\): Evaluation of Approximation with absolute values

<table>
<thead>
<tr>
<th>N</th>
<th>N!</th>
<th>In N!</th>
<th>N ln N – N</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.63 x 10^6</td>
<td>15.1</td>
<td>13.02</td>
<td>13.8%</td>
</tr>
<tr>
<td>50</td>
<td>3.04 x 10^64</td>
<td>148.4</td>
<td>145.6</td>
<td>1.88%</td>
</tr>
<tr>
<td>100</td>
<td>9.33 x 10^{157}</td>
<td>363.7</td>
<td>360.5</td>
<td>0.88%</td>
</tr>
<tr>
<td>150</td>
<td>5.71 x 10^{262}</td>
<td>605.0</td>
<td>601.6</td>
<td>0.56%</td>
</tr>
</tbody>
</table>

Calculators often overheat at 200!, which is all right since clearly result are converging. In thermodynamics, we are often dealing very large N (i.e., of the order of Avagadro’s number) and for these values Stirling’s approximation is excellent.

References


Contributors and Attributions

- SklogWiki
- Stefan Franzen (North Carolina State University)