Stirling's approximation is named after the Scottish mathematician James Stirling (1692-1770). In confronting statistical problems we often encounter factorials of very large numbers. The factorial \( N! \) is a product \( N(N-1)(N-2)\ldots(2)(1) \). Therefore, \( \ln(N!) \) is a sum

\[
\left[ \ln N! \right. = \ln 1 + \ln 2 + \ln 3 + \ldots + \ln N = \sum_{k=1}^{N} \ln k. \label{1}
\]

where we have used the property of logarithms that \( \log(abc) = \log(a) + \log(b) + \log(c) \). The sum is shown in figure below.

Using Euler-MacLaurin formula one has

\[
\sum_{k=1}^{N} \ln k = \int_1^{N} \ln x \, dx + \sum_{k=1}^{p} \frac{B_{2k}}{2k(2k-1)} \left( \frac{1}{n^{2k-1}} - 1 \right) + R \label{2}
\]

where \( B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, \ldots \) are the Bernoulli numbers, and \( R \) is an error term which is normally small for suitable values of \( p \).

Then, for large \( N \),

\[
\ln N! \sim \int_1^{N} \ln x \, dx \approx N \ln N - N. \label{3}
\]

after some further manipulation one arrives at (apparently Stirling's contribution was the prefactor of \( \sqrt{2\pi} \))

\[
N! = \sqrt{2\pi N} \; N^{N} e^{-N} e^\lambda \label{4}
\]

where

\[
\dfrac{1}{12N+1} < \lambda_N < \frac{1}{12N}. \label{5}
\]

The sum of the area under the blue rectangles shown below up to \( N \) is \( \ln N! \). As you can see the rectangles begin to closely approximate the red curve as \( m \) gets larger. The area under the curve is given the integral of \( \ln x \).

\[
\sum_{m=1}^{N} \ln m \approx \int_1^{N} \ln x \, dx \approx N \ln N - N. \label{6}
\]

To solve the integral use integration by parts

\[
\int u \, dv = uv - \int v \, du \label{7A}
\]
Here we let $u = \ln x$ and $dv = dx$. Then $v = x$ and $du = \frac{1}{x}dx$.

$$\int_0^N \ln x \, dx = x \ln x|_0^N - \int_0^N x \frac{1}{x} \, dx \label{7B}$$

Notice that $(x/x = 1)$ in the last integral and $(x \ln x)$ is 0 when evaluated at zero, so we have

$$\int_0^N \ln x \, dx = N \ln N - \int_0^N dx \label{8}$$

Which gives us Stirling’s approximation: $(\ln N! = N \ln N – N)$. As is clear from the figure above Stirling’s approximation gets better as the number $N$ gets larger (Table \(\PageIndex{1}\)).

Table \(\PageIndex{1}\): Evaluation of Approximation with absolute values

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N!$</th>
<th>$\ln N!$</th>
<th>$N \ln N - N$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$3.63 \times 10^6$</td>
<td>15.1</td>
<td>13.02</td>
<td>13.8%</td>
</tr>
<tr>
<td>50</td>
<td>$3.04 \times 10^{64}$</td>
<td>148.4</td>
<td>145.6</td>
<td>1.88%</td>
</tr>
<tr>
<td>100</td>
<td>$9.33 \times 10^{157}$</td>
<td>363.7</td>
<td>360.5</td>
<td>0.88%</td>
</tr>
<tr>
<td>150</td>
<td>$5.71 \times 10^{262}$</td>
<td>605.0</td>
<td>601.6</td>
<td>0.56%</td>
</tr>
</tbody>
</table>

Calculators often overheat at $200!$, which is all right since clearly result are converging. In thermodynamics, we are often dealing very large $N$ (i.e., of the order of Avagadro’s number) and for these values Stirling’s approximation is excellent.

References


Contributors and Attributions

- SklogWiki
- Stefan Franzen (North Carolina State University)