The uncertainty principle comes about fundamentally from the commuting properties of any two quantum operators. So for any two observables, \(\langle A \rangle\) and \(\langle B \rangle\), then the generalized uncertainty principle states that \(\langle \Delta_A \rangle^2 \langle \Delta_B \rangle^2 \geq (1/4)|\langle [A,B] \rangle|^2\) where the \(\langle \rangle\) denote expectation values, the \(\langle \Delta_Q \rangle\) denotes the variance in the \(\langle Q \rangle\) operator, and \(\langle [A,B] = AB - BA \rangle\) is the commutator. Now this definition shows you that there are really only two choices, either the observables commute or they don't. If they commute \(\langle [A,B] = 0 \rangle\) then there are no restrictions on how accurately we may determine them, if they do not commute then the generalized uncertainty principle holds. This definition shows that it is possible (and relatively trivial) to create operators that are not conjugate, but where they do not commute and below are three examples.

**Position and Momentum**

If we examine a particle traveling freely described by a quantum-mechanical plane wave, \(\psi = Ae^{ikx}\) if the particle is traveling in the \(+x\) direction. We have seen that the momentum of this particle can be calculated exactly using the momentum operator

\[
i\hbar \dfrac{\partial}{\partial x} e^{ikx} = \hbar k e^{ikx} = Pe^{ikx}
\]

If we now ask where this particle is we examine the probability density of the particle \(\psi^* \psi = A^2\). Since the square of A has no x-dependence we have no information regarding the position of the particle. In other words, if we know the momentum of the particle exactly, the position is completely unknown.

Suppose we assume that the particle can be located in space and has a Gaussian probability distribution. Then instead of a constant prefactor \(\langle A \rangle\) we would have a Gaussian prefactor and the wavefunction becomes

\[
\psi = B e^{-(x - x_0)^2/2} e^{ikx}
\]

Note that the Gaussian function expressed in terms of \((x - x_0)\) signifies a Gaussian centered at \(x_0\). This distribution in space implies a distribution of momenta since

\[
\psi(p) \int_{-\infty}^{\infty} e^{-(x - x_0)^2/2} e^{ipx/h} dx
\]

Which is a Fourier transform of the two conjugate variables position and momentum. Note how the relationship between these two conjugate variables implies a quantitative relationship between their distributions. To calculate \(\langle \psi(p) \rangle\) we need to complete the square. In order to demonstrate this we will calculate a simpler distribution. Imagine that our particle is found centered at the origin so \(x_0 = 0\). We can then solve the integral

\[
\psi(k) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{ikx} dx
\]

By multiplying by a term equal to \(e^{-k^2/2}\) inside and \(e^{k^2/2}\) outside the integral.

\[
\psi(k) = e^{-k^2/2} \int_{-\infty}^{\infty} e^{-x^2/2} e^{ikx} e^{k^2/2} dx
\]

Thus, we have

\[
\psi(k) = e^{-k^2/2} \int_{-\infty}^{\infty} e^{-(x-ik)^2/2} dx = \sqrt{\pi} e^{-k^2/2}
\]
Note that in fact this gives a moment distribution since \( (k = p/\hbar) \). Note also that substituting \( (x - x_0) \) for \( (x) \) will not change this result. This result shows that the Fourier transform of a Gaussian is also a Gaussian (in the space of the conjugate variable). This result is quite useful for quantifying the relationships between conjugate variables.

How does this result give us an uncertainty principle? We now include a Gaussian standard deviation that we will call \( \langle \Delta(x) \rangle \) and one for momentum that we will call \( \langle \Delta(k) \rangle \).

\[
\Psi(k) = e^{-k^2/2\Delta_k^2} \int_{-\infty}^{\infty} e^{-x^2/2\Delta_x^2}e^{ikx}e^{k^2/2\Delta_k^2}dx
\]

The Fourier transform then has the form

\[
\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(k) e^{-ikx} dk
\]

which implies that \( \langle \Delta(x) \Delta(k) \rangle = 1 \) in order for \( \langle e^{ikx} \rangle \) to conform to the definition of a Fourier transform.

This in turn implies that \( \langle \Delta(x) \Delta(p) \rangle = \hbar \). The relationship described here implies that the linear momentum and position along the x-axis do not commute, \( \langle x, p_x \rangle \neq 0 \). In fact, \( \langle x, p_x \rangle = -i\hbar \).

### Angular Momenta

The argument that we have made for linear momentum also applies to the angular momentum of a particle. For example, consider a particle traversing a circular trajectory. In this case we have seen that the wave function is \( \psi = e^{im\phi} \). This implies azimuthal angular momenta of \( m \hbar \). For each of these momentum states if we now ask where the particle is on the circle we find.

\[
\int_0^{2\pi} \psi^* \psi d\phi
\]

for

\[
\psi = \frac{1}{\sqrt{2\pi}} e^{im\phi}
\]

So that the probability of finding the particle in any region of space is \( \langle 1/2|\phi| \rangle \). There is no \( \langle \phi \rangle \) dependence and hence the location of the particle on the circle is completely undetermined. We can therefore understand that the same type of relationship exists between angular momenta and angular position as exists for linear momenta and linear position. The angular momentum relationships can also be expressed in terms of the total angular momentum and the azimuthal angular momentum. That is that we can know both \( \langle L_z \rangle \) and \( \langle L \rangle \) simultaneously, but not \( \langle L_x \rangle \) or \( \langle L_y \rangle \) and \( \langle L_z \rangle \) simultaneously.

\[
\langle [L_z, L] = 0 \rangle
\]

although the individual components do not commute:

- \( \langle [L_z, L_x] = i\hbar L_y \rangle \)
- \( \langle [L_x, L_y] = i\hbar L_z \rangle \)
- \( \langle [L_y, L_z] = i\hbar L_x \rangle \)
Energy and Time

It is important that the same type of conjugate variable relationship exists between the energy $e$ and time $t$ such that the uncertainty in the energy times the uncertainty in the time is of the order of Planck's constant as well. $\Delta_e \Delta_t = \hbar$ exactly analogous to the relationship between momentum and position given above. The mathematical form of the of the Fourier transform is identical if

$$k \rightarrow \omega$$

and

$$x \rightarrow t.$$ 

The latter relationship is important since it implies that the lifetime of a state results in an energy width or the duration of a laser pulse implies a certain spectral width. If we use the properties of Gaussians shown above for the position/momentum conjugate variables we can calculate a spectral width for a given time duration. We do this in terms of Joules and wave numbers. The wave number unit is particularly valuable since spectra are reported in cm$^{-1}$.

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