This tutorial presents three pictures of the 4s state of the one-dimensional hydrogen atom using its position, momentum and phase-space representations.

The energy operator for the one-dimensional hydrogen atom in atomic units is:
\[
\frac{-1}{2} \frac{d^2}{dx^2} \Theta - \frac{1}{x} \Theta
\]

The position 4s wave function is:
\[
\begin{matrix}
\Psi (x) = \frac{x}{4} \left( 1 - \frac{3}{4} x + \frac{1}{8} x^2 - \frac{1}{192} x^3 \right) \text{exp} \left( \frac{-x}{4} \right) & \int_0^{\infty} \Psi (x)^2 \text{dx} = 1 \end{matrix}
\]

The 4s energy is -0.03125 E\text{h}.

\[
\frac{-1}{2} \frac{d^2}{dx^2} \Psi (x) - \frac{1}{x} \Psi (x) = E \Psi (x) \text{solve, } E \rightarrow \frac{-1}{32} = -0.03125
\]

The momentum wave function is generated by the following Fourier transform of the coordinate wave function.
\[
\Phi (p) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \text{exp}(-i p x) \Psi (x) \text{dx} \rightarrow (-2) 2^{\frac{1}{2}} \frac{64 i p^3 - 48 p^2 - 12 i p + 1}{(4 i p + 1)^5 \pi^{\frac{1}{2}}}
\]
The Wigner function (phase-space representation) for the hydrogen atom 4s state is generated using the momentum wave function.

\[
\text{W(x, p)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\Phi - \left( p + \frac{s}{2} \right)} \exp(-i s x) \Phi \left( p - \frac{s}{2} \right) ds
\]

The Wigner distribution is displayed graphically.

\[
\begin{matrix}
N = 100 & i = 0 .. N & x_i = \frac{50i}{N} & j = 0 .. N & p_j = -2 + \frac{4j}{N} & \text{Wigner}_{i, j} = \text{W} \left( x_i, p_j \right)
\end{matrix}
\]

Just as for the 2s and 3s states, the Wigner distribution for the 4s state takes on negative values.