The Linear Superposition

Unpolarized light consists of photons of all possible polarization angles. A photon polarized at an angle \( \theta \) with respect to the vertical can be written as a linear superposition of a vertically polarized photon, \( |v\rangle \), and a horizontally polarized photon, \( |h\rangle \). \( |v\rangle \) and \( |h\rangle \) are the polarization basis states.

\[
|\theta\rangle = |v\rangle \cos \theta + |h\rangle \sin \theta
\]

From the figure above it can be seen that the projection of \( |\theta\rangle \) onto \( |v\rangle \) and \( |h\rangle \) are \( \cos \theta \) and \( \sin \theta \) respectively.

\[
|\langle \theta |v\rangle^2 = \cos^2 \theta
\]

The probability that a photon polarized at an angle \( \theta \) will pass a vertical polarizer is

\[
|\langle \theta | \theta\rangle |^2 = \cos^2 \theta
\]

By integrating this function over all possible angles we find that half of the incident light passes through a vertical polarizer. See the figure below.

\[
\frac{\int_0^{2\pi} \cos^2 \theta d\theta}{2\pi} = 0.5
\]
The photons that pass the vertical polarizer are now vertically polarized. That is they are eigenfunctions of that measurement operator.

The probability that a vertically polarized photon will pass a second filter that is vertically polarized is one.

\[ | \langle v | v \rangle |^2 = \cos^2 0^\circ = 1 \]

The probability that a vertically polarized photon will pass a second filter that is horizontally polarized is zero.

\[ | \langle h | v \rangle |^2 = \cos^2 90^\circ = 0 \]

The vertically polarized photon can be written as linear superposition of any other set of orthogonal basis states, for example \( |45^\circ\rangle\) and \( |-45^\circ\rangle\).

\[ |v\rangle = |45^\circ\rangle \langle 45^\circ | v \rangle + |-45^\circ\rangle \langle -45^\circ | v \rangle \]

\[ |v\rangle = |45^\circ\rangle 0.707 + |-45^\circ\rangle 0.707 \]

Now if a 45° polarizer is inserted in between the vertical and horizontal polarizers photons get through the horizontal polarizer that stopped them previously (see the last figure).

Here's the explanation. The probability that a vertically polarized photon will get through a polarizer oriented at an angle of 45° is 0.5. See the figure below.

\[ | \langle 45^\circ | v \rangle |^2 = \cos^2 45^\circ = 0.5 \]
Now the photon is in the state of $|45^\circ\rangle$ which can be written as a linear superposition of $|v\rangle$ and $|h\rangle$.

\[|45^\circ\rangle = |v\rangle \cos 45^\circ + |h\rangle \sin 45^\circ\]

\[|45^\circ\rangle = |v\rangle 0.707 + |h\rangle 0.707\]

Thus, the probability that this photon will pass the final horizontally oriented polarizer is

\[|\langle H | 45^\circ \rangle|^2 = \sin^2 45^\circ = 0.5\]

See the figure below.

In this last figure the intensity of the light emerging from the final horizontal polarizing filter can be calculated compactly as

\[\frac{\int_{0}^{2\pi} |\langle h | 45^\circ \rangle \langle 45^\circ | v \rangle \langle v | \theta \rangle|^2 \, d\theta}{2\pi} = \frac{1}{8}\]
The term inside the integral is the probability that a photon with polarization $\theta$ will pass through the three filters. This expression is integrated over all values of $\theta$. 