A Gaussian function, \( \exp(-\alpha r^2) \), is proposed as a trial wavefunction in a variational calculation on the hydrogen atom. Determine the optimum value of the parameter \( \alpha \) and the ground state energy of the hydrogen atom. Use atomic units: \( h = 2\pi \), \( m_e = 1 \), \( e = -1 \).

\[
\Phi(r, \beta) := \left( \frac{2 \beta}{\pi} \right)^{\frac{3}{4}} \exp(-\beta r^2)
\]

\[
T = \frac{-1}{2r} \frac{d^2}{dr^2} (r \Phi(r, \beta))
\]

\[
V = \frac{1}{r}
\]

\[
\int_{0}^{\infty} 4\pi r^2 dr
\]

a. Demonstrate the wave function is normalized.

\[
\int_{0}^{\infty} \Psi(r, \beta)^2 4\pi r^2 dr \rightarrow 1
\]

b. Evaluate the variational integral.

\[
E(\beta) := \int_{0}^{\infty} \Psi(r, \beta) \left( \frac{-1}{2r} \frac{d^2}{dr^2} (r \Psi(r, \beta)) \right) 4\pi r^2 dr
\]

\[
\beta := 1 \quad \beta := \text{Minimize}(E, \beta) \quad \beta = 0.283 
E(\beta) = -0.424
\]

d. The exact ground state energy for the hydrogen atom is \(-0.5\) \( E_h \). Calculate the percent error.

\[
\frac{-0.5 - E(\beta)}{-0.5} = 15.117
\]

e. The differences between the Gaussian and Slate type wavefunctions are illustrated with the surface plots shown below.

\[
\begin{align*}
N &:= 50 
b &:= 5 \quad i := 0..N \quad j := 0..N \quad (y_{-i}) := -b + \frac{2bi}{N} \quad (x_{-j}) := -b + \frac{2bj}{N} \\
\text{Gauss}_{i,-j} &:= \left( \frac{2 \beta}{\pi} \right)^{\frac{3}{4}} \exp\left[-\beta \left( (x_{-j})^2 + (y_{-i})^2 \right)\right] \\
\text{Slater}_{i,-j} &:= \frac{1}{\sqrt{\pi}} \exp\left[-\sqrt{ (x_{-j})^2 + (y_{-i})^2 }\right]
\end{align*}
\]
f. These wavefunctions can also be compared to their radial distribution functions:

\[ r := 0, .1 .. 6 \]

\[ G(r) := \left( \frac{2 \beta}{\pi} \right)^{\frac{3}{4}} \exp\left( -\beta r^2 \right) \]

\[ S(r) := \frac{1}{\sqrt{\pi}} \exp\left( -r \right) \]