The quantum harmonic oscillator is the quantum analog of the classical harmonic oscillator and is one of the most important model systems in quantum mechanics. This is due in part to the fact that an arbitrary potential curve \( V(x) \) can usually be approximated as a harmonic potential at the vicinity of a stable equilibrium point. Furthermore, it is one of the few quantum-mechanical systems for which an exact, analytical solution exists. Solving other potentials typically require either approximations or numerical approaches to identify the corresponding eigenstates and eigenvalues (i.e., wavefunctions and energies).

![Figure](PagelIndex{1})

Figure \((\text{HCl})\) molecule is really an anharmonic oscillator. Here \((\text{HCl})\) is vibrating at energy level \((E_3)\). \((D_0)\) is dissociation energy here, \((r_0)\) bond length, \((V)\) potential energy. Energy is expressed in wavenumbers. The hydrogen chloride molecule is attached to the coordinate system to show bond length changes on the curve. Image used with permission from Wikipedia.

A general potential energy \((V(x))\) curve for a molecular vibration can be expanded as a Taylor series (Figure \((\text{PagelIndex}2)\))

\[
\begin{align*}
V(x) &= V(x_0) + \left. \frac{d V(x)}{d x} \right|_{x_0} (x - x_0) + \left. \frac{1}{2!} \frac{d^2 V(x)}{d x^2} \right|_{x_0} (x - x_0)^2 + \ldots + \left. \frac{1}{n!} \frac{d^n V(x)}{d x^n} \right|_{x_0} (x - x_0)^n \\
V(x) &= \frac{1}{2} k x^2 + \frac{1}{6} \gamma x^3
\end{align*}
\]

where \(V(x_0) = 0\), \((k)\) is the harmonic force constant (harmonic term), and \((\gamma)\) is the first (i.e., cubic) anharmonic term. It is important to note that this approximation is only good for \((x)\) near \((x_0)\), and that \((x_0)\) stands for the equilibrium bond distance.
Almost all diatomics have experimentally determined $\frac{d^2 V}{d x^2}$ for their lowest energy states. $\text{H}_2$, $\text{Li}_2$, $\text{O}_2$, $\text{N}_2$, and $\text{F}_2$ have had terms up to $\langle n < 10 \rangle$ determined of Equation $\ref{5.3.1}$.

Adding anharmonic perturbations to the harmonic oscillator (Equation $\ref{5.3.2}$) better describes molecular vibrations. Anharmonic oscillation is described as the deviation of a system from harmonic oscillation, or an oscillator not oscillating in simple harmonic motion. Anharmonic oscillation is described as the restoring force is no longer proportional to the displacement. Figure $\PageIndex{1}$ shows the general potential with (numerically) calculated energy levels ($\langle E_0 \rangle$, $\langle E_1 \rangle$ etc.). $\langle D_o \rangle$ is the dissociation energy, which is different from the well depth $\langle D_e \rangle$. These vibrational energy levels of this plot can be calculated using the harmonic oscillator model (i.e., Equation $\ref{5.3.1}$) with the Schrödinger equation and have the general form

$$E_v = \left(v + \frac{1}{2}\right)v_e - \left(v + \frac{1}{2}\right)^2v_e x_e + \left(v + \frac{1}{2}\right)^3v_e y_e + \text{higher terms} \label{5.3.7}$$

where $\langle v \rangle$ is the vibrational quantum number and $\langle x_e \rangle$ and $\langle y_e \rangle$ are the first and second anharmonicity constants, respectively.
The $v = 0$ level is the vibrational ground state. Because this potential is less confining than a parabola used in the harmonic oscillator, the energy levels become **less widely** spaced at high excitation (Figure \(\PageIndex{1}\); top of potential).

Limitations of the Harmonic Oscillator Model for Molecular Vibrations

The harmonic oscillation is a great approximation of a molecular vibration, but has key limitations:

- Due to equal spacing of energy, all transitions occur at the **same** frequency (i.e. single line spectrum). However experimentally many lines are often observed (called overtones).
- The harmonic oscillator does not predict bond dissociation; you cannot break it no matter how much energy is introduced.

Morse Potentials are better Approximations of Vibrational Motion

A more powerful approach than just "patching up" the harmonic oscillator solution with anharmonic corrections is to adopt a different potential \(\langle V(x) \rangle\). One such approach is the Morse potential, named after physicist Philip M. Morse, and a better approximation for the vibrational structure of the molecule than the harmonic oscillator because it explicitly includes the effects of **bond breaking** and accounts for the anharmonicity of real bonds (Figure \(\PageIndex{4}\)).

![Morse Potential](image.jpg)

Figure \(\PageIndex{4}\): The Morse potential (blue) and harmonic oscillator potential (green). Unlike the energy levels of the harmonic oscillator potential, which are evenly spaced by $\hbar \omega$, the Morse potential level spacing decreases as the energy approaches the dissociation energy. The dissociation energy $D_e$ is larger than the true energy required for dissociation $D_0$ due to the zero point energy of the lowest ($v = 0$) vibrational level. Image used with permission from Wikipedia.

The Morse Potential is a good approximation to \(\langle V(x) \rangle\) and is best when looking for a general formula for all $\langle x \rangle$ from 0 to $\langle \infty \rangle$, not just applicable for the local region around the $\langle x_0 \rangle$:

$$\langle V(x) \rangle = D(1-e^{-\beta (x - x_0)})^2 \langle \text{label}(5.3.8) \rangle$$
with \((V(x = x_0) = 0)\) and \((V(x = \infty) = D)\).

The Morse Potential (Figure\PageIndex{4})) approaches zero at infinite \((r_e)\) and equals \((-D_e)\) at its minimum (i.e. \((r=r_e)\)). It clearly shows that the Morse potential is the combination of a short-range repulsion term (small \((r)\) values) and a long-range attractive term (large \((r)\) values).

Solving the Schrödinger Equation with the Morse Potential (Equation \ref{5.3.8}) is not trivial, but can be done analytically.

\[
\hat{H} |\psi \rangle = E_n |\psi \rangle
\]

with

\[
\begin{align}
\hat{H} &= \hat{T} + \hat{V} \\
&= \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + D(1-e^{-\beta(x-x_0)})^2
\end{align}
\label{5.3.9}
\]

The solutions and energies for the Morse potential will not be used in this course and will not be discussed in more detail.

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**Contributors**

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