Learning Objectives

• Explore the basis of the oscillatory solutions to the wave equation
• Understand the consequences of boundary conditions on the possible solutions
• Rationalize how satisfying boundary conditions forces quantization (i.e., only solutions with specific wavelengths exist)

The boundary conditions for the string held to zero at both ends argue that $u(x,t)$ collapses to zero at the extremes of the string (Figure \(\PageIndex{1}\)).

\[K = - p^2 \tag{2.3.1}\]

The general solution to differential equations of the form of Equation \ref{2.3.2} is

\[X(x) = A e^{ix} + B e^{-ix} \tag{2.3.3}\]

Example \(\PageIndex{1}\)

Verify that Equation \ref{2.3.3} is the general form for differential equations of the form of Equation \ref{2.3.2}.
which when substituted with Equation \(\text{(ref{2.3.1})}\) give
\[
X(x) = A e^{ipx} + B e^{-ipx} \tag{2.2.4}
\]

Expand the complex exponentials into trigonometric functions via Euler formula \((e^{i \theta} = \cos \theta + i \sin \theta)\)
\[
X(x) = A \left[ \cos (px) + i \sin (px) \right] + B \left[ \cos (px) - i \sin (px) \right] \tag{2.3.5}
\]

collecting like terms
\[
X(x) = (A + B) \cos (px) + i(A - B) \sin (px) \tag{2.3.6}
\]

Introduce new *complex* constants \((c_1 = A + B)\) and \((c_2 = i(A - B))\) so that the general solution in Equation \(\text{(ref{2.3.6})}\) can be expressed as oscillatory functions
\[
X(x) = c_1 \cos (px) + c_2 \sin (px) \tag{2.3.7}
\]

Now let's apply the boundary conditions from Equation 2.2.7 to determine the constants \((c_1)\) and \((c_2)\). Substituting the first boundary condition \((X(x=0)=0)\) into the general solutions of Equation \(\text{(ref{2.3.7})}\) results in
\[
X(x=0) = c_1 \cos (0) + c_2 \sin (0) = 0, \text{ at } x=0 \, \text{label(2.3.8a)}
\]
\[
c_1 = 0 \, \text{label(2.3.8b)}
\]
\[
c_1 = 0 \, \text{label(2.3.8c)}
\]

and substituting the second boundary condition \((X(x=L)=0)\) into the general solutions of Equation \(\text{(ref{2.3.7})}\) results in
\[
X(x=L) = c_1 \cos (pL) + c_2 \sin (pL) = 0, \text{ at } x=L \, \text{label(2.3.9)}
\]
we already know that \((c_1=0)\) from the first boundary condition so Equation \(\text{(ref{2.3.9})}\) simplifies to
\[
c_2 \sin (pL) = 0 \, \text{label(2.3.10)}
\]

Given the properties of sines, Equation \(\text{(ref{2.3.9})}\) simplifies to
\[
pL = n\pi \, \text{label(2.3.11)}
\]
with \((n=0)\) is the *trivial solution* that we ignore so \((n = 1, 2, 3...)).
\[
p = \frac{np}{L} \, \text{label(2.3.12)}
\]

Substituting Equations \(\text{(ref{2.3.12})}\) and \(\text{(ref{2.3.8c})}\) into Equation \(\text{(ref{2.3.7})}\) results in
\[
X(x) = c_2 \sin \left(\frac{np x}{L}\right) \, \text{right label(2.3.13)}
\]
which can simplify to
\[ X(x) = c_2 \sin \left( \omega x \right) \label{2.3.14} \]

with

\[ \omega = \frac{n \pi}{L} \]

A similar argument applies to the other half of the ansatz (\( T(t) \)).

Exercise \( \PageIndex{1} \)

Given two traveling waves:

\[ \psi_1 = \sin{(c_1 x + c_2 t)} \quad \text{and} \quad \psi_2 = \sin{(c_1 x - c_2 t)} \]

a. Find the wavelength and the wave velocity of \( \psi_1 \) and \( \psi_2 \)

b. Find the following and identify nodes:

\[ \psi_+ = \psi_1 + \psi_2 \quad \text{and} \quad \psi_- = \psi_1 - \psi_2 \]

**Solution a:**

\( \psi_1 \) is a sin function. At every integer \( n \) \( \pi \) where \( n=0, \pm 1, \pm 2, \ldots \), a sin function will be zero. Thus, \( \psi_1 = 0 \) when \( c_1 x + c_2 t = \pi n \). Solving for \( x \), while ignoring trivial solutions:

\[ x = \frac{n \pi - c_2 t}{c_1} \]

The velocity of this wave is:

\[ \frac{dx}{dt} = -\frac{c_2}{c_1} \]

Similarly for \( \psi_2 \). At every integer \( n \) \( \pi \) where \( n=0, \pm 1, \pm 2, \ldots \), a sin function will be zero. Thus, \( \psi_2 = 0 \) when \( c_1 x - c_2 t = \pi n \). Solving for \( x \), for \( \psi_2 \):

\[ x = \frac{n \pi + c_2 t}{c_1} \]

The velocity of this wave is:

\[ \frac{dx}{dt} = \frac{c_2}{c_1} \]

The wavelength for each wave is twice the distance between two successive nodes. In other words,

\[ \lambda = 2(x_n - x_{n-1}) = \frac{2 \pi}{c_1} \]

**Solution b:**

Find \( \psi_+ = \psi_1 + \psi_2 \quad \text{and} \quad \psi_- = \psi_1 - \psi_2 \).

\[ \begin{align*}
\psi_+ & = \sin{(c_1 x + c_2 t)} + \sin{(c_1 x - c_2 t)} \\
& = \sin{(c_1 x)} \cos{(c_2 t)} + \cancel{\cos{(c_1 x)}} \cos{(c_2 t)} + \cancel{\cos{(c_1 x)}} \sin{(c_2 t)} + \cancel{\sin{(c_1 x)}} \sin{(c_2 t)} \\
\psi_- & = \sin{(c_1 x + c_2 t)} - \sin{(c_1 x - c_2 t)} \\
& = \sin{(c_1 x)} \cos{(c_2 t)} - \cancel{\cos{(c_1 x)}} \cos{(c_2 t)} - \cancel{\cos{(c_1 x)}} \sin{(c_2 t)} + \cancel{\sin{(c_1 x)}} \sin{(c_2 t)} 
\end{align*} \]
\[
\sin(c_1 t) + \sin (c_1 x) \cos (c_2 t) - \cancel{\cos(c_1 x) \sin(c_1 t)} \quad \text{[4pt]}  
= 2\sin (c_1 x) \cos (c_2 t) \quad \text{[align*]}
\]

This should have a node at every \( x = n \pi / c_1 \)

And

\[
\begin{align*}
\psi_- &= \sin (c_1 x + c_2 t) - \sin (c_1 x - c_2 t) \\
&= \cancel{\sin (c_1 x) \cos (c_2 t)} + \cos(c_1 x) \sin(c_1 t) - \cancel{\sin (c_1 x) \cos (c_2 t)} + \cos(c_1 x) \sin(c_1 t) \\
&= 2\cos (c_1 x) \sin (c_2 t) \quad \text{[align*]}
\end{align*}
\]