Here we review the derivation of the vector potential for the plane wave in free space. We begin with Maxwell's equations (SI):

\[
\begin{align}
\overline{\nabla} \cdot \overline{B} &= 0 \label{6.78} \\
\overline{\nabla} \cdot \overline{E} &= \rho / \varepsilon_0 \label{6.79} \\
\overline{\nabla} \times \overline{E} &= -\frac{\partial \overline{B}}{\partial t} \label{6.80} \\
\overline{\nabla} \times \overline{B} &= \mu_0 \overline{J} + \varepsilon_0 \mu_0 \frac{\partial \overline{E}}{\partial t} \label{6.81}
\end{align}
\]

Here the variables are: \(\overline{E}\), electric field; \(\overline{B}\), magnetic field; \(\overline{J}\), current density; \(\rho\), charge density; \(\varepsilon_0\), electrical permittivity; \(\mu_0\), magnetic permeability. We are interested in describing \(\overline{E}\) and \(\overline{B}\) in terms of a vector and scalar potential, \(\overline{A}\) and \(\varphi\).

### Vector Fields

Next, let's review some basic properties of vectors and scalars. Generally, vector field \(\overline{F}\) assigns a vector to each point in space. The divergence of the field

\[
\overline{\nabla} \cdot \overline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \label{6.82}
\]

is a scalar. For a scalar field \(\phi\), the gradient

\[
\nabla \phi = \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z} \label{6.83}
\]

is a vector for the rate of change at one point in space. Here

\[
\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{r}^2
\]

are unit vectors. Also, the curl

\[
\overline{\nabla} \times \overline{F} = \left| \begin{array}{ccc}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{array} \right|
\]

is a vector whose \(x\), \(y\), and \(z\) components are the circulation of the field about that component. Some useful identities from vector calculus that we will use are

\[
\overline{\nabla} \cdot (\overline{\nabla} \times \overline{F}) = 0 \label{6.85}
\]

\[
\nabla \times (\nabla \phi) = 0 \label{6.86}
\]

\[
\nabla \times (\overline{\nabla} \times \overline{F}) = \overline{\nabla} (\overline{\nabla} \cdot \overline{F}) - \overline{\nabla}^2 \overline{F} \label{6.87}
\]
Gauge Transforms

We now introduce a vector potential \( \overline{A}(\overline{r},t) \) and a scalar potential \( \varphi(\overline{r},t) \), which we will relate to \( \overline{E} \) and \( \overline{B} \). Since

\[
\overline{\nabla} \cdot \overline{B} = 0
\]

and

\[
\overline{\nabla} (\overline{\nabla} \times \overline{A}) = 0,
\]

we can immediately relate the vector potential and magnetic field

\[
\overline{B} = \overline{\nabla} \times \overline{A} \tag{6.88}
\]

Inserting this into Equation \ref{6.80} and rewriting, we can relate the electric field and vector potential:

\[
\overline{\nabla} \times \left[ \overline{E} + \frac{\partial \overline{A}}{\partial t} \right] = 0 \tag{6.89}
\]

Comparing Equations \ref{6.89} and \ref{6.86} allows us to state that a scalar product exists with

\[
\overline{E} = \frac{\partial \overline{A}}{\partial t} - \nabla \varphi \tag{6.90}
\]

So summarizing our results, we see that the potentials \( \overline{A} \) and \( \varphi \) determine the fields \( \overline{B} \) and \( \overline{E} \):

\[
\overline{B}(\overline{r},t) = \overline{\nabla} \times \overline{A}(\overline{r},t) \tag{6.91}
\]

\[
\overline{E}(\overline{r},t) = -\overline{\nabla} \varphi(\overline{r},t) - \frac{\partial}{\partial t} \overline{A}(\overline{r},t) \tag{6.92}
\]

We are interested in determining the classical wave equation for \( \overline{A} \) and \( \varphi \). Using Equation \ref{6.91}, differentiating Equation \ref{6.92}, and substituting into Equation \ref{6.81}, we obtain

\[
\overline{\nabla} \times (\overline{\nabla} \times \overline{A}) + \varepsilon_0 \mu_0 \left( \frac{\partial^2 \overline{A}}{\partial t^2} + \overline{\nabla} \frac{\partial \varphi}{\partial t} \right) = \mu_0 \overline{J} \tag{6.93}
\]

Using Equation \ref{6.87},

\[
\left[ -\overline{\nabla}^2 \overline{A} + \varepsilon_0 \mu_0 \frac{\partial^2 \overline{A}}{\partial t^2} \right] + \overline{\nabla} \left( \overline{\nabla} \cdot \overline{A} + \varepsilon_0 \mu_0 \frac{\partial \varphi}{\partial t} \right) = \overline{\mu}_0 \overline{J} \tag{6.94}
\]

From Equation \ref{6.90}, we have

\[
\overline{\nabla} \cdot \overline{E} = -\frac{\partial}{\partial t} \left( \overline{\nabla} \cdot \overline{A} \right) - \overline{\nabla} \cdot \overline{\mu}_0 \overline{J} \tag{6.95}
\]
\( \nabla^2 \varphi \) label{6.95}

and using Equation \ref{6.79},

\[
\frac{-\partial \overline{V} \cdot \overline{A}}{\partial t} - \overline{\nabla}^2 \varphi = \rho / \varepsilon_0 \label{6.96}
\]

Notice from Equations \ref{6.91} and \ref{6.92} that we only need to specify four field components (..., \&xyz?) to determine all six \(\overline{A}\) and \(\overline{B}\) components. But \(E\) and \(B\) do not uniquely determine \(A\) and \(\varphi\). So we can construct \(A\) and \(\varphi\) in any number of ways without changing \(\overline{E}\) and \(\overline{B}\). Notice that if we change \(A\) by adding \(\delta A\) where \(\delta\) is any function of \(r\) and \(t\), this will not change \(\overline{B}\). It will change \(\overline{E}\) by \(\delta \overline{E} = \delta \varphi \), but we can change \(\varphi\) to \(\varphi + \delta \varphi\) and \(\overline{B}\) will both be unchanged. This property of changing representation (gauge) without changing \(\overline{E}\) and \(\overline{B}\) is \textbf{gauge invariance}. We can define a gauge transformation with

\[
\overline{A}'(\overline{r}, t) = \overline{A}(\overline{r}, t) + \overline{\nabla} \cdot \chi(\overline{r}, t) \label{6.97}
\]

\[
\varphi'(\overline{r}, t) = \varphi(\overline{r}, t) - \frac{\partial}{\partial t} \chi(\overline{r}, t) \label{6.98}
\]

Up to this point, \(A\) and \(\varphi\) are undetermined. Let's choose \(\varphi\) such that:

\[
\overline{\nabla} \cdot \overline{A} + \varepsilon_0 \mu_0 \frac{\partial \varphi}{\partial t} = 0 \label{6.99}
\]

which is known as the \textbf{Lorentz condition}. Then from Equation \ref{6.93}:

\[
\frac{-\nabla^2 \overline{A}}{\varepsilon_0 \mu_0} - \frac{\partial^2 \overline{A}}{\partial t^2} = \mu_0 \overline{J} \label{6.100}
\]

The right hand side of this equation can be set to zero when no currents are present. From Equation \ref{6.96}, we have:

\[
\varepsilon_0 \mu_0 \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\varepsilon_0} \label{6.101}
\]

Equations \ref{6.100} and \ref{6.101} are wave equations for \(\overline{A}\) and \(\varphi\). Within the Lorentz gauge, we can still arbitrarily add another \(\varphi\); it must only satisfy Equation \ref{6.99}. If we substitute Equations \ref{6.97} and \ref{6.98} into Equation \ref{6.101}, we see

\[
\nabla^2 \chi - \varepsilon_0 \mu_0 \frac{\partial^2 \chi}{\partial t^2} = 0 \label{6.102}
\]

So we can make further choices/constraints on \(A\) and \(\varphi\); as long as it obeys Equation \ref{6.102}. We now choose \(\varphi = 0\), the \textbf{Coulomb gauge}, and from Equation \ref{6.99} we see

\[
\overline{\nabla} \cdot \overline{A} = 0 \label{6.103}
\]
So the wave equation for our vector potential when the field is far currents ($J = 0$)) is

$$-
abla^2 A + \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \quad \text{(6.104)}$$

The solutions to this equation are plane waves:

$$A = A_0 \sin(\omega t - k \cdot r + \alpha) \quad \text{(6.105)}$$

where $\alpha$ is a phase. $k$ is the wave vector which points along the direction of propagation and has a magnitude

$$k^2 = \omega^2 \mu_0 \varepsilon_0 = \omega^2 / c^2 \quad \text{(6.106)}$$

Since

$$\nabla \cdot A = 0$$

(Equation ref(6.103)),

$$-k \cdot A_0 \cos(\omega t - k \cdot r + \alpha) = 0$$

therefore

$$k \cdot A_0 = 0 \quad \text{(6.107)}$$

So the direction of the vector potential is perpendicular to the direction of wave propagation ($k \perp A_0$). From Equations ref(6.91) and ref(6.92), we see that for $\varphi = 0$:

$$E = -\frac{\partial A}{\partial t} = -\omega A_0 \cos(\omega t - k \cdot r + \alpha) \quad \text{(6.108)}$$

$$B = \nabla \times A = -\left( k \times A_0 \right) \cos(\omega t - k \cdot r + \alpha) \quad \text{(6.109)}$$

Here the electric field is parallel with the vector potential, and the magnetic field is perpendicular to the electric field and the direction of propagation ($k \perp E \perp B$)). The Poynting vector describing the direction of energy propagation is

$$S = \varepsilon_0 c^2 (E \times B)$$

and its average value, the intensity, is

$$I = \langle S \rangle = \frac{1}{2} \varepsilon_0 c E_0^2.$$