The equation of motion for the density matrix follows naturally from the definition of \( \rho \) and the time-dependent Schrödinger equation.

\[
\begin{align}
\frac{\partial \rho}{\partial t} &= \frac{\partial}{\partial t} [ | \psi \rangle \langle \psi | ] \\
&= \left[ \frac{\partial}{\partial t} | \psi \rangle \right] \langle \psi | + | \psi \rangle \left[ \frac{\partial}{\partial t} \langle \psi | \right] \\
&= \frac{-i}{\hbar} H | \psi \rangle \langle \psi | + \frac{i}{\hbar} | \psi \rangle \langle \psi | H. \label{4.13}
\end{align}
\]

Equation \ref{4.14} is the \textbf{Liouville-Von Neumann equation}. It is isomorphic to the Heisenberg equation of motion, since \( \rho \) is also an operator. The solution to Equation \ref{4.14} is

\[
\rho ( t ) = U \rho ( 0 ) U ^ {\dagger} \label{4.15}
\]

This can be demonstrated by first integrating Equation \ref{4.14} to obtain

\[
\rho ( t ) = \rho ( 0 ) - \frac{i}{\hbar} \int_{0}^{t} d \tau [ H ( \tau ) , \rho ( \tau ) ] \label{4.16}
\]

If we expand Equation \ref{4.16} by iteratively substituting into itself, the expression is the same as when we substitute

\[
U = \exp \left\{ + \right\} \left[ - \frac{i}{\hbar} \int_{0}^{t} d \tau H ( \tau ) \right] \label{4.17}
\]

into Equation \ref{4.15} and collect terms by orders of \( \langle H(\tau) \rangle \).

Note that Equation \ref{4.15} and the cyclic invariance of the trace imply that the time-dependent expectation value of an operator can be calculated either by propagating the operator (Heisenberg) or the density matrix (Schrödinger or interaction picture):

\[
\begin{aligned}
\langle \hat{A} ( t ) \rangle &= \text{Tr} [ \hat{A} \rho ( t ) ] \\
&= \text{Tr} [ \hat{A} U \rho _{0} U ^{\dagger} ] \\
&= \text{Tr} [ \hat{A} ( t ) \rho _{0} ]
\end{aligned}
\]

\[
\begin{align}
\rho _{nm} ( t ) &= \langle n | \rho ( t ) | m \rangle \\
&= \langle n | U | \psi _{0} \rangle \langle \psi _{0} | U ^{\dagger} | m \rangle \\
&= e^{-i \omega _{nm} ( t - t_{0} )} \rho _{nm} ( t_{0} ) \label{4.19}
\end{align}
\]

From this we see that populations, \( \rho _{nm} ( t ) = \rho _{nm} ( t _{0} ) \), are time-invariant, and coherences oscillate at the energy splitting \( \langle \omega _{nm} \rangle \).