# Worksheet 6B Solutions

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1. The Hamiltonian for the harmonic oscillator is:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + \frac{kx^2}{2} \tag{1}$$

We are asked to use the trial function

$$|\psi(x)\rangle = \frac{1}{1+\beta x^2} \tag{2}$$

to find an approximate energy:

$$E_\phi = \frac{\langle\phi|\,\hat{H}\,|\phi\rangle}{\langle\phi|\phi\rangle} \tag{3}$$

   (a) We evaluate the denominator of the expression, $\langle\phi|\phi\rangle$ via:

$$\langle\phi|\phi\rangle = \int_{-\infty}^{\infty}\frac{dx}{(1+\beta x^2)^2} = \frac{\pi}{2\beta^{1/2}} \tag{4}$$

   (b) The function is smooth as both it and its first derivative are finite. The limits of the integral are negative to positive infinity.

   (c) The wavefunction $|\phi\rangle$ is not normalized. Normalization is inherent in the equation (1), with the denominator being the normalization constant squared.

2. We are required to evaluate the numerator, $\langle\phi|\,\hat{H}\,|\phi\rangle$

   (a) This requires us to evaluate the separate terms:

$$\langle\phi| -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + \frac{kx^2}{2}\,|\phi\rangle = \langle\phi| -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\,|\phi\rangle + \langle\phi|\,\frac{kx^2}{2}\,|\phi\rangle \tag{5}$$

$$\langle\phi| -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\,|\phi\rangle = -\frac{\hbar^2}{2\mu}\left(\int_{-\infty}^{\infty}\frac{2\beta dx}{(1+\beta x^2)^2} + \int_{-\infty}^{\infty}\frac{4\beta x^2 dx}{(1+\beta x^2)^4}\right) \tag{6}$$

$$\langle\phi|\,\frac{kx^2}{2}\,|\phi\rangle = \frac{k}{2}\int_{-\infty}^{\infty}\frac{x^2 dx}{(1+\beta x^2)} \tag{7}$$

   We will make use of the following integrals:

$$\int_{-\infty}^{\infty}\frac{dx}{(1+\beta x^2)^2} = \frac{\pi}{2\beta^{1/2}} \tag{8}$$

$$\int_{-\infty}^{\infty}\frac{x^2 dx}{(1+\beta x^2)^2} = \frac{\pi}{2\beta^{3/2}} \tag{9}$$

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+\beta x^2)^4} = \frac{\pi}{16\beta^{3/2}} \tag{10}$$

Using relations (8) and (10) to simplify (6) and relation (9) to simplify (7) the numerator terms become:

$$\langle\phi| - \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + \frac{kx^2}{2}|\phi\rangle = -\frac{\hbar^2}{2\mu}\left(2\beta\Big(\frac{\pi}{2\beta^{1/2}}\Big) + 4\beta\Big(\frac{\pi}{16\beta^{3/2}}\Big)\right) + \frac{k}{2}\left(\frac{\pi}{16\beta^{3/2}}\right) \tag{11}$$

Putting together results (11) and (4), so multiplying the above terms by $\frac{2\beta^{1/2}}{\pi}$ we finally have:

$$E_\phi = \frac{\langle\phi|\hat{H}|\phi\rangle}{\langle\phi|\phi\rangle} = -\frac{\hbar^2}{\mu}\left(\beta + \frac{1}{8\beta}\right) + \frac{k}{32\beta} = -\frac{\hbar^2}{\mu}\beta + \left(\frac{k}{32} - \frac{\hbar^2}{8\mu}\right)\frac{1}{\beta} \tag{12}$$