Worksheet 9A Solutions

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1. System of Equations

   Consider the following system of equations:

$$a_{11}x + a_{12}y = d_1 \tag{1}$$

$$a_{11}x + a_{12}y = d_2 \tag{2}$$

where a_{ij} and d_i are scalars.

   (a) How can you manipulate these two expressions to get an expression that only depends on x ?

   We can multiply each equation by a constant, such that the resulting coefficients in front of either x or y are the same. We can then subtract one scaled equation from the other scaled and eliminate a variable.

   (b) Using this method, what expression results when you solve for x?

   Multiplying (1) by a_{22}, multiplying (2) by a_{12} and then subtracting them we obtain:

$$x = \frac{a_{22}d_1 - a_{12}d_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{3}$$

   (c) What is the corresponding expression for y?

$$y = \frac{a_{21}d_1 - a_{11}d_2}{a_{12}a_{21} - a_{11}a_{22}} \tag{4}$$

2. Denominators

   (a) What is the value of the determinant $\begin{vmatrix} 5 & 6 \\ 11 & 20 \end{vmatrix}$?

   $\begin{vmatrix} 5 & 6 \\ 11 & 20 \end{vmatrix} = 5 * 20 - 6 * 11 = 34$

   (b) What is the explicit form of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$?

   $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ac - bd$

   ?

   (c) What is the general explicit form of $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$?

   $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$

3. Cofactors and Expansions

(a) For the matrix, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, what is the cofactor for a_{12}?

The cofactor for a_{12} is $-\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

(b) How could this construct be useful? How could it simplify problems?

It is useful because it allows us to reduce large determinants to expressions in terms of smaller ones. Alternatively, we would have to resort to the combinatoric definition of a determinant and sum up many long terms. It can simply problems by strategic choice of the row or column we would choose cofactors from.

(c) How would you use cofactors to solve the problem, $\begin{vmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{vmatrix} = 0$?

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{vmatrix} = x\begin{vmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & x & 1 \\ 0 & 1 & x \end{vmatrix} = x\left(x\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & x \end{vmatrix}\right) - 1\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = 0$$

This simplifies further down to:

$$x(x(x^2-1)-x)-(x^2-1) = x(x(x^2-2))-x^2+1 = x^2(x^2-2)-x^2+1 = x^2(x^2-3)+1 = x^4-3x^2+1 = 0$$